Lecture 15. Null spaces, column spaces, and row spaces

<u>Def</u> Let A be a matrix.

(1) Its <u>null space</u> $\mathrm{Nul}(A)$ is the solution set of the equation $A\vec{x} = \vec{0}$.

(2) Its <u>column space</u> $\mathrm{Col}(A)$ is the span of the columns

(3) Its <u>row space</u> $\mathrm{Row}(A)$ is the span of the rows

<u>Prop</u> Let A be an $m \times n$ matrix.

(1) $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

(2) $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$.

(3) $\mathrm{Row}(A)$ is a subspace of $\mathbb{R}^n$.

<u>pf</u> (1) • zero vector : $\vec{0}$ in $\mathrm{Nul}(A)$ $(A\vec{0} = \vec{0})$

• closed under addition :

$\vec{u}$ and $\vec{v}$ in $\mathrm{Nul}(A)$ $\Longrightarrow A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$

$\Longrightarrow \vec{u}+\vec{v}$ in $\mathrm{Nul}(A)$

• closed under scalar multiplication :

$\vec{u}$ in $\mathrm{Nul}(A) \Longrightarrow A(c\vec{u}) = cA\vec{u} = c \cdot \vec{0} = \vec{0}$ for any $c \in \mathbb{R}$

$\Longrightarrow c\vec{u}$ in $\mathrm{Nul}(A)$ for any $c \in \mathbb{R}$

Hence $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^n$

(2) $\mathrm{Col}(A)$ is the <u>span</u> of the columns

$\Longrightarrow \mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$

(3) $\mathrm{Row}(A)$ is the <u>span</u> of the rows

$\Longrightarrow \mathrm{Row}(A)$ is a subspace of $\mathbb{R}^n$

Def Given a matrix A, its <u>transpose</u> A^T is the matrix whose columns are given by the rows of A (in the same order)

e.g. $A = \begin{bmatrix} 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 5 \\ 3 & 1 & 2 & 0 \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$

Note (1) If A is an $m \times n$ matrix, A^T is an $n \times m$ matrix.

(2) $\text{Row}(A) = \text{Col}(A^T)$

(3) $(A^T)^T = A$

* There are many interesting properties of transpose that we will never use in Math 313

e.g. $(A+B)^T = A^T + B^T$, $(AB)^T = B^T A^T$

Prop Let A be an $m \times n$ matrix.

(1) A vector $\vec{v} \in \mathbb{R}^m$ lies in $\text{Col}(A) \iff A\vec{x} = \vec{v}$ has a solution

(2) A vector $\vec{w} \in \mathbb{R}^n$ lies in $\text{Row}(A) \iff A^T\vec{y} = \vec{w}$ has a solution

pf (1) Let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ be the columns of A

$\vec{v}$ lies in $\text{Col}(A)$

$\iff$ $\vec{v}$ lies in the span of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$

$\iff$ $\vec{v} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$ for some $x_1, x_2, \cdots, x_n \in \mathbb{R}$

$\iff$ $\vec{v} = A\vec{x}$ has a solution

(2) $\vec{w}$ lies in $\text{Row}(A) = \text{Col}(A^T)$

$\underset{\text{by (1)}}{\iff}$ $\vec{w} = A^T\vec{y}$ has a solution

Ex Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \end{bmatrix}$$

(1) Determine whether Nul(A) contains $\vec{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

Sol $A\vec{u} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Longrightarrow$ Nul(A) $\boxed{\text{contains } \vec{u}}$

(2) Determine whether Col(A) contains $\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Sol We consider the equation $A\vec{x} = \vec{v}$

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 2 & 3 & 5 & 2 \end{array}\right]}_{A \quad \vec{v}} \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 2 \end{array}\right]$$

no leading 1's in the last column

The equation has a solution.

$\Longrightarrow$ Col(A) $\boxed{\text{contains } \vec{v}}$

(3) Determine whether Row(A) contains $\vec{w} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Sol We consider the equation $A^T\vec{y} = \vec{w}$

$$\underbrace{\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 4 & 5 & 2 \end{array}\right]}_{A^T \quad \vec{w}} \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

a leading 1 in the last column

The equation has no solutions.

$\Longrightarrow$ Row(A) $\boxed{\text{does not contain } \vec{w}}$

Ex If possible, express each set either as Nul(A) or Col(A) for a suitable matrix A.

(1) The set of all vectors of the form $\begin{bmatrix} a-3b \\ 2a+b \end{bmatrix}$ with $a, b \in \mathbb{R}$

Sol We may write $\begin{bmatrix} a-3b \\ 2a+b \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\Rightarrow$ The set is the span of $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

$\Rightarrow$ The set is $\boxed{\mathrm{Col}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}}$

(2) The set of all points (x, y, z) in $\mathbb{R}^3$ with $x+y=2z$ and $y=2x+3z$.

Sol The set is given by the solutions of the linear system

$$\begin{cases} x+y-2z=0 \\ 2x-y+3z=0 \end{cases}$$

$\Rightarrow$ The set is $\boxed{\mathrm{Nul}\begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 3 \end{bmatrix}}$

Note We may visualize the set as the intersection of the planes given by $x+y-2z=0$ and $2x-y+3z=0$

(3) The set of all points (x, y) in $\mathbb{R}^2$ with $y=x+2$

Sol The set is not a vector space as it does not contain the zero point $(0,0)$.

$\Rightarrow$ The set is $\boxed{\text{not Nul(A) or Col(A) for any matrix A}}$